Fermionic Casimir effect with helix boundary condition

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In this paper, we consider the fermionic Casimir effect under a new type of space-time topology using the concept of quotient topology. The relation between the new topology and that in Ref. [3, 4] is something like that between a Möbius strip and a cylindric. We obtain the exact results of the Casimir energy and force for the massless and massive Dirac fields in the $(D + 1)$-dimensional space-time. For both massless and massive cases, there is a Z_2 symmetry for the Casimir energy. To see the effect of the mass, we compare the result with that of the massless one and we found that the Casimir force approaches the result of the force in the massless case when the mass tends to zero and vanishes when the mass tends to infinity.

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I. INTRODUCTION

Casimir's computation of the force between two neutral, parallel conducting plates[1] originally inspired much theoretical interest as macroscopic manifestation of quantum fluctuation of the field in vacuum. However, the Casimir force arises not only in the presence of material boundaries, but also in spaces with non-Euclidean topology[2] and the attractive or repulsive nature of the Casimir force may depend on the topology of spacetime. The simplest example of the Casimir effect of topological origin is the scalar field on a flat manifold with topology of a circle S^1. The topology of S^1 causes the periodicity condition $\phi(t, 0) = \phi(t, C)$ for a Hermitian scale field $\phi(t, x)$, where C is the circumference of S^1, imposed on the wave function which is of the same kind as those due to boundary and resulting in an attractive Casimir force. Similarly, the antiperiodic conditions can be drawn on a Möbius strip and bring about the repulsive Casimir force as a result. Recently, the topology of the helix boundary conditions is investigated in Ref.[3]. We find that the Casimir force is very much like the force on a spring that obeys the Hooke's law in mechanics. However, in this case, the force comes from a quantum effect, so we would like to call this structure a quantum spring. The force is attractive in both massless and massive scalar cases for this structure[4].

It is worth noting that the concept of quotient topology is very useful for concrete application. We consider a surjective mapping f from a topological space X onto a set Y. The quotient topology on Y with respect to f is given in [5]. Surjective mapping can be easily obtained when we use the equivalence classes of some equivalence relation $\sim$. Thus, we let $X/\sim$ denote the set of equivalence classes and define $f : X \rightarrow X/\sim$ by $f(x) = [x]$ the equivalence class containing x. $X/\sim$ with the quotient topology is called to be obtained from X by topological identification. For example, if we take the unit square $I^2 = \{(x_1, x_2); 0 \leq x_1, x_2 \leq 1\}$ in $\mathbb{R}^2$ with the induced topology and define an equivalence relation $\sim$ on I^2 by

$$(x_1, x_2) \sim (x'_1, x'_2) \Leftrightarrow (x_1, x_2) = (x'_1, x'_2)$$
$$\text{or } \{x_1, x'_1\} = \{0, 1\} \quad \text{and} \quad x_2 = x'_2 \tag{1}$$

then $I^2/\sim$ with the quotient topology is homomorphic to the cylinder C

$$C = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 = 1, |x_3| \leq 1\} \tag{2}$$

The boundary condition $\phi(t, 0, x_2) = \phi(t, 1, x_2)$ can be drawn on a cylinder.

The ζ-function regularization procedure is a powerful and elegant technique for the Casimir effect. Rigorous extension of the proof of Epstein ζ-function regularization has been discussed in [6]. Vacuum polarization in the background of on string was first considered in [7]. The generalized ζ-function has many interesting applications, e.g., in the piecewise string [8–10]. Similar analysis has been applied to cosmology entropy [11], p-branes [12], rectangular

cavity [13, 14], or pistons [15–19]. Casimir effect for a fermionic field is of interest in considering, for example, the structure of proton in particle physics [20, 21] while the thermofield dynamics of Casimir effect for Dirac fermion fields has been studied [22] and recently for Majorana fermion fields [23]. The Casimir piston of fermion is also studied [24]. And the fermionic Casimir effect in the presence of compact dimensions has been recently considered in Refs. [25] and [26].

In this paper, we consider the fermionic Casimir effect under a new type of space-time topology using the concept of quotient topology. The relation between the new helix topology and that we use in [3, 4] is something like that between a Möbius strip and a cylindric. We obtain the exact results of the Casimir energy and Casimir force for the massless and massive Dirac fields in the $(D+1)$-dimensional space-time.

II.   THE VACUUM ENERGY DENSITY FOR A FERMIONIC FIELD

As mentioned in Sec. 1, the Casimir effect arises not only in presence of material boundaries, but also in space with nontrivial topology. We consider topological space X as follows

$$X = \bigcup_{\mathtt{u} \in \Lambda''} \{C_0 + \mathtt{u}\} \tag{3}$$

in $\mathcal{M}^{D+1}$ with the induced topology and define an equivalence relation $\sim$ on X by

$$(x^1, x^2) \sim (x^1 - 2a, x^2 + 2h) \tag{4}$$

then $X/\sim$ with the quotient topology is homomorphic to helix topology. Here, Λ'' and unit cylinder-cell C_0 [3, 4] are

$$\Lambda'' = \{ \, n(\mathtt{e}_2 - \mathtt{e}_1) \mid n \in \mathcal{Z} \, \} \, . \tag{5}$$

and

$$
\begin{aligned}
C_0 \;=\; & \left\{ \sum_{i=0}^{D} x^i \mathtt{e}_i \mid 0 \le x^1 < a, -h \le x^2 < 0, \right. \\
& \left. -\infty < x^0 < \infty, -\frac{L}{2} \le x^T \le \frac{L}{2} \right\},
\end{aligned} \tag{6}
$$

where $T = 3, \cdots, D$. Next, we discuss what are referred to as anti-helix conditions imposed on a field ψ,

$$\psi(t, x^1 + a, x^2, x^T) = -\psi(t, x^1, x^2 + h, x^T), \tag{7}$$

where the field returns to the same value after traveling distances $2a$ at the x^1-direction and $2h$ at the x^2-direction. It is notable that a spinor wave function is anti-helix and takes its initial value after traveling distances $2a$ and $2h$ respectively. In other words, the anti-helix conditions are imposed on the field, which returns to the same field value $\psi(t, x^1 + 2a, x^2, x^T) = \psi(t, x^1, x^2 + 2h, x^T)$ only after two round trips. Therefore, the boundary condition (7) can be induced by $X/\sim$ with the quotient topology.

In calculations on the Casimir effect, extensive use is made of eigenfunctions and eigenvalues of the corresponding field equation. A spin-1/2 field $\psi(t, x^\alpha, x^T)$ defined in the $(D+1)$-dimensional flat space-time satisfies the Dirac equation:

$$i\gamma^\mu \partial_\mu \psi - m_0 \psi = 0 \,, \tag{8}$$

where $\alpha = 1, 2; T = 3, \cdots, D; \mu = (t, \alpha, T)$ and m_0 is the mass of the Dirac field. γ^μ are $N \times N$ Dirac matrices with $N = 2^{[(D+1)/2]}$ where the square brackets mean the integer part of the enclosed expression. We will assume that these matrices are given in the chiral representation:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k^+ & 0 \end{pmatrix}, k = 1, 2, \cdots, D \tag{9}$$

with the relation $\sigma_\mu \sigma_\nu^+ + \sigma_\nu \sigma_\mu^+ = 2\delta_{\mu\nu}$. Under the boundary condition (7), the solutions of the field can be presented as

$$\psi^{(+)} = \mathcal{N}^{(+)} e^{-i\omega t} \begin{pmatrix} e^{i(k_x x + k_z z + k_T x^T)} \varphi_{(\alpha)} \\ -i\boldsymbol{\sigma}^+ \cdot \boldsymbol{\nabla} e^{i(k_x x + k_z z + k_T x^T)} \varphi_{(\alpha)} / (\omega + m_0) \end{pmatrix}, \tag{10}$$

and

$$\psi^{(-)} = \mathcal{N}^{(-)} e^{i\omega t} \begin{pmatrix} i\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} e^{i(k_x x + k_z z + k_T x^T)} \chi_{(\alpha)}/(\omega + m_0) \\ e^{i(k_x x + k_z z + k_T x^T)} \chi_{(\alpha)} \end{pmatrix}, \tag{11}$$

where $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_D)$, $x^1 = x, x^2 = z$ and $\mathcal{N}^{(\pm)}$ is a normalization factor, and $\varphi_{(\alpha)}, \chi_{(\alpha)}$ are one-column constant matrices having $2^{[(D+1)/2]} - 1$ rows with the element $\delta_{\alpha\beta}, \alpha, \beta = 1, \cdots, 2^{[(D+1)/2]} - 1$.

From Eqs. (8)-(11), we have

$$\begin{aligned} \omega_n^2 &= k_T^2 + k_x^2 + \left(-\frac{2\pi(n + \frac{1}{2})}{h} + \frac{k_x}{h} a \right)^2 + m_0^2 \\ &= k_T^2 + k_z^2 + \left(\frac{2\pi(n + \frac{1}{2})}{a} + \frac{k_z}{a} h \right)^2 + m_0^2. \end{aligned} \tag{12}$$

Here, k_x and k_z satisfy

$$ak_x - hk_z = 2\left(n + \frac{1}{2}\right)\pi, (n = 0, \pm 1, \pm 2, \cdots). \tag{13}$$

In the ground state (vacuum), each of these modes contributes an energy of $\omega_n/2$. The energy density of the field in $(D+1)$-dimensional space-time is thus given by

$$\begin{aligned} E^D = &- \frac{N}{2a} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \\ &\times \sum_{n=-\infty}^{\infty} \sqrt{k_T^2 + k_z^2 + \left(\frac{2\pi(n + \frac{1}{2})}{a} + \frac{k_z}{a} h \right)^2 + m_0^2} \end{aligned} \tag{14}$$

where we have assumed $a \neq 0$ without losing generalities.

Eq.(14) can be rewritten as

$$\begin{aligned} E^D = &- \frac{N}{2a\sqrt{\gamma}} \int \frac{d^{D-1}u}{(2\pi)^{D-1}} \\ &\times \sum_{n=-\infty}^{\infty} \sqrt{u^2 + \left(\frac{2\pi(n + \frac{1}{2})}{a\sqrt{\gamma}} \right)^2 + m_0^2}, \end{aligned} \tag{15}$$

where

$$\gamma \equiv 1 + \frac{h^2}{a^2}. \tag{16}$$

Using the mathematical identity,

$$\int_{-\infty}^{\infty} f(u) d^{D-1}u = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)} \int_0^{\infty} u^{D-2} f(u) du, \tag{17}$$

one can express the vacuum energy density as

$$E^D = \frac{2^{[(D+1)/2]-(D+1)} \Gamma\left(-\frac{D}{2}\right)}{\pi^{\frac{D}{2}} a\sqrt{\gamma}} \sum_{n=-\infty}^{\infty} \left[\left(\frac{2\pi(n + \frac{1}{2})}{a\sqrt{\gamma}} \right)^2 + m_0^2 \right]^{\frac{D}{2}} \tag{18}$$

It is seen from Eq. (15) that the expression for the vacuum energy in the case of helix boundary conditions can be obtained from the corresponding expression in the case of standard boundary condition $\psi(t, x^1 + a, x^2, x^T) = -\psi(t, x^1, x^2, x^T)$ by making the change $a \to a\sqrt{\gamma} = \sqrt{a^2 + h^2}$. The topological fermionic Casimir effect in toroidally compactified space-times has been recently investigated in Ref. [25] for non-helix boundary conditions including general phases. In the limiting case $h = 0$, our result of Eq. (18) is a special case of general formulas from Ref.[25].

III. THE CASE OF MASSLESS FIELD

For a massless Dirac field, that is , in the case of $m_0 = 0$, the energy density in Eq.(18) is reduced to

$$E_0^D = \frac{2^{[(D+1)/2]}\pi^{\frac{D}{2}}}{a^{D+1}\gamma^{\frac{D+1}{2}}}\Gamma\left(-\frac{D}{2}\right)\zeta(-D,\frac{1}{2}). \tag{19}$$

where $\zeta(-D,\frac{1}{2})$ is the Hurwitz-Riemann ζ function. Using the relation

$$\zeta(s,\frac{1}{2}) = (2^s - 1)\zeta(s), \tag{20}$$

and the reflection formula

$$\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{s-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{21}$$

the energy density can be regularized to be

$$E_{R,0}^D = 2^{[(D+1)/2]}\left(2^{-D} - 1\right)\frac{\Gamma\left(\frac{D+1}{2}\right)\zeta(D+1)}{\pi^{\frac{D+1}{2}}(a^2 + h^2)^{\frac{D+1}{2}}}. \tag{22}$$

The Casimir force on the x direction is

$$\begin{aligned}
F_{a,0} &= -\frac{\partial E_{R,0}^D}{\partial a} = 2^{[(D+1)/2]}\left(2^{-D} - 1\right) \\
&\times \frac{(D+1)\Gamma\left(\frac{D+1}{2}\right)\zeta(D+1)}{\pi^{\frac{D+1}{2}}}\frac{a}{(a^2 + h^2)^{\frac{D+3}{2}}}
\end{aligned} \tag{23}$$

It is obvious that the energy density is negative and the force is attractive. Furthermore, the force has a maximum value

$$\begin{aligned}
F_{a,0}^{max} &= 2^{[(D+1)/2]}\left(2^{-D} - 1\right)\frac{(D+1)\Gamma\left(\frac{D+1}{2}\right)\zeta(D+1)}{\pi^{\frac{D+1}{2}}h^{D+2}} \\
&\times \sqrt{\frac{(D+2)^{D+2}}{(D+3)^{D+3}}}
\end{aligned} \tag{24}$$

at $a = \frac{h}{\sqrt{D+2}}$. The results for $F_{h,0}$ are similar to those of $F_{a,0}$ because of the symmetry between a and h.

Fig. 1 is the illustration of the behavior of the Casimir force on x direction in $D = 3$ dimension. The curves correspond to $h = 0.9, 1.0, 1.1, 1.2$ respectively. It is clearly seen that the attractive Casimir force decreases with h increasing and the maximum value of the force $-\frac{35\sqrt{5}\pi^2}{1944h^5}$ appears at $a = \frac{h}{\sqrt{5}}$.

Fig. 2 is the illustration of the behavior of the Casimir force on x direction in different dimensions. The curves correspond to $D = 2, 3, 4, 5$ respectively. We take $h = 2.5$ in this figure. It is clearly seen that the value of a where the maximum value of the force is achieved gets smaller with D increasing.

IV. THE CASE OF MASSIVE FIELD

For a massive Dirac field, to regularize the series in Eq. (18) we use the Chowla-Selberg formula directly [27]

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}\left[\frac{1}{2}a(n+c)^2 + b\right]^{-s} \\
&= \frac{(2\pi)^{\frac{1}{2}}b^{\frac{1}{2}-s}}{\sqrt{a}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} + \frac{2^{\frac{s}{2}+\frac{1}{4}+2}\pi^s b^{-\frac{s}{2}+\frac{1}{4}}}{\sqrt{a}\Gamma(s)} \\
&\times \sum_{n=1}^{\infty}\cos(2\pi nc)\left(\frac{n^2}{a}\right)^{\frac{s}{2}-\frac{1}{4}}K_{\frac{1}{2}-s}\left(2\pi n\sqrt{\frac{2b}{a}}\right)
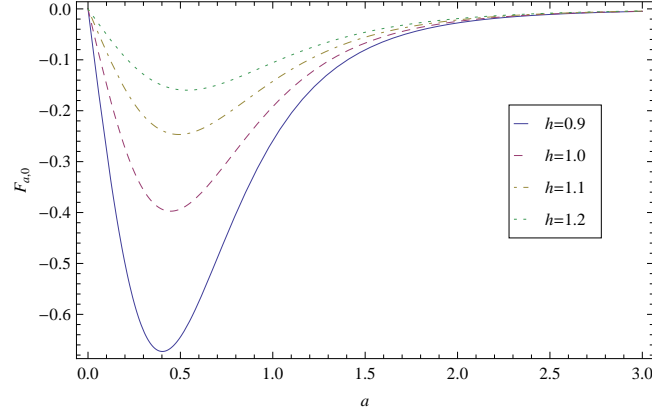\end{aligned} \tag{25}$$

FIG. 1: The fermionic Casimir force on the x direction *vs.* a in $D = 3$ dimension for different h. The Casimir force decreases with h increasing and the maximum value of the force $-\frac{35\sqrt{5}\pi^2}{1944h^5}$ appears at $a = \frac{h}{\sqrt{5}}$.
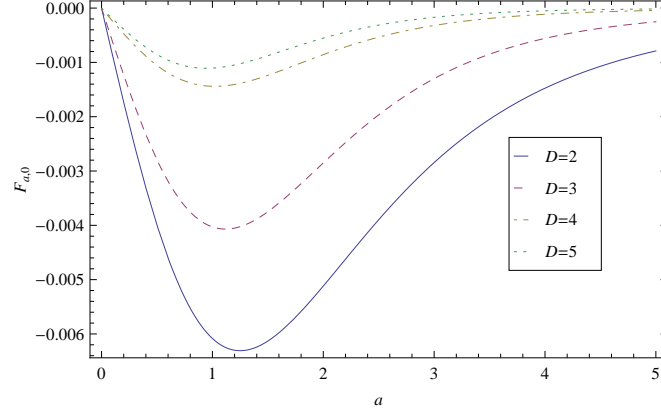

FIG. 2: The fermionic Casimir force on the x direction *vs.* a in different dimensions. Here we take $h = 2.5$. It is clearly seen that the value of a where the maximum value of the force is achieved gets smaller with D increasing.

where $K_\nu(z)$ is the modified Bessel function. Note that in the renormalization procedure, the vacuum energy in a flat space-time with trivial topology should be renormalized to zero, that is, in the expression for the renormalized vacuum energy the term corresponding to the first term in the right hand side of Eq. (25) should be ommited. Finally, the Casmir energy has the expression as follows

$$
\begin{aligned}
E_{R,m_0}^D &= 2^{[(D+3)/2]} \left(\frac{m_0}{2\pi\sqrt{a^2+h^2}} \right)^{\frac{D+1}{2}} \\
&\times \sum_{n=1}^{\infty} \cos(\pi n) n^{-\frac{D+1}{2}} K_{\frac{D+1}{2}} \left(n m_0 \sqrt{a^2+h^2} \right)
\end{aligned}
\tag{26}
$$

For $\nu > 0$ and $z \to 0$, the Bessel function has the asymptotic expression $K_\nu(z) \to \frac{2^{\nu-1}\Gamma(\nu)}{z^\nu}$, so it is not difficult to find that when $m_0 \to 0$, the Casimir energy recover the result of the massless case.

Using the relation $K_\nu'(z) = \frac{\nu}{z} K_\nu(z) - K_{\nu+1}(z)$ where $K_\nu'(z) = dK_\nu(z)/dz$, we have the Casimir force

$$F_{a,m_0} = \frac{2^{[(D+3)/2]} m_0 a \left((m_0 a)^2 + (m_0 h)^2\right)^{\frac{D+1}{4}}}{(2\pi)^{\frac{D+1}{2}} (a^2 + h^2)^{\frac{D+2}{2}}}$$

$$\times \sum_{n=1}^{\infty} \cos(\pi n) n^{-\frac{D-1}{2}} K_{\frac{D+3}{2}} \left(n m_0 \sqrt{a^2 + h^2}\right)$$

$$(27)$$

We study numerically the behavior of the Casimir force on x direction as a function of a for different h and D. We find that the Casimir force is still attractive and it has a maximum value similarly to massless case. Because the precise way the Casimir force varies as the mass changes is worth studying, we give the numerical results in Figs. 3 and 4 for the ratio of the Casimir force in massive and massless cases varying with the mass for different h and D.

Fig.3 is the illustration of the ratio of the Casimir force in massive case to that in massless case varying with the mass in $D = 3$ dimension. The curves correspond to $a = 1$ and $h = 0.1, 1, 2, 3$ respectively. Fig.4 is the illustration of the ratio of the Casimir force in massive case to that in massless case varying with the mass for different dimensions. The curves correspond to $a = 1, h = 0.1$ and $D = 2, 3, 4, 5$ respectively. It is clearly seen from the two figures that the Casimir force decreases with m_0 increasing, and it approaches zero when m_0 tends to infinity. The plots also tell us that for a given mass, the ratio decreases with h increasing but it increases with D increasing.
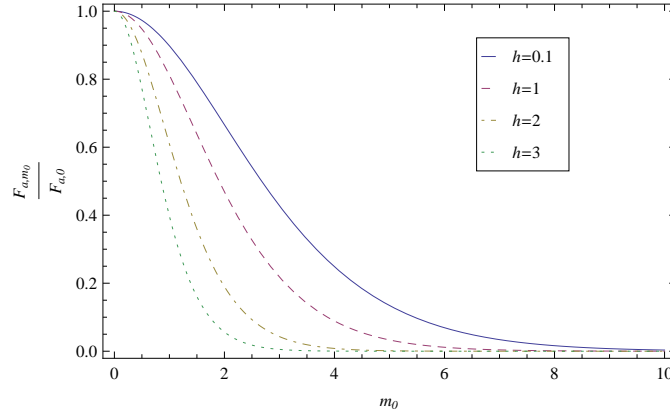

FIG. 3: The ratio of the Casimir force in massive case to that in massless case varying with the mass for different h in $D = 3$ dimension. The curves correspond to $a = 1$ and $h = 0.1, 1, 2, 3$ respectively.
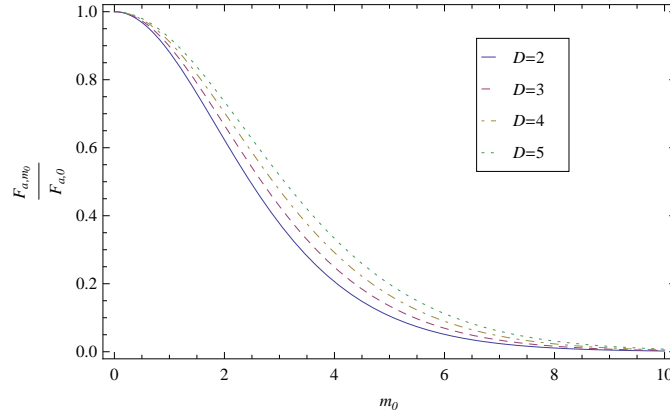

FIG. 4: The ratio of the Casimir force in massive case to that in massless case varying with the mass for different dimensions. We take $a = 1$ and $h = 0.1$.

V. CONCLUSION

The Casimir force arises not only in the presence of material boundaries, but also in spaces with non-Euclidean topology and the attractive or repulsive nature of the Casimir force may depend on the topology of space-time. We have studied the Casimir effect of a Dirac fermionic field under the helix topology. We saw that the fermionic Casimir force is attractive under the helix boundary condition.

Another interesting character is that the fermionic Casimir force has a maximum value and the force decreases with D increasing. Equally interesting is that there is a Z_2 symmetry of $a \leftrightarrow h$. Besides, when the mass of fermion tends to zero, the Casimir force approaches the result of the force in massless case and when the mass tends to infinity, the Casimir force for a massive field goes to zero.

In recent years much attention has been paid to the possibility that a universe could have non-trivial topology. As is known that the Casimir effect can apply to the cosmology with extra dimensions, the effect of the quantum spring in higher dimensional cosmology is worth considering and we will study it in our further work.

Acknoledgments

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